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Asymptotic properties for Volterra integro-dynamic systems

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Abstract. Using the resolvent matrix, a comparison principle and a useful equivalent system, we investigate the asymptotic behavior of linear Volterra integro-dynamic systems on time scales.

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1 Introduction

Infectious diseases have long been recognized as a major cause of mortality in human and other populations. The spread of an infectious disease involves not only disease-related factors such as the infectious agent, mode of transmission, latent period, infectious period, but also social, demographic and geographic factors [18]. Most of the work in the literature in modeling infectious disease epidemics is mathematically inspired and based on integro-differential systems [15].

Classical topics in the qualitative theory of integro-differential equations are asymptotic equivalence and asymptotic behavior of systems [7, 12]. Two systems of integro-differential equations are said to be asymptotically equivalent if, corresponding to each solution of one system, there exists a solution of the other system such that the difference between these two solutions tends to zero. If we know that two systems are asymptotically equivalent, and if we also know the asymptotic behavior of the solutions of one of the system, then we can obtain information about the asymptotic behavior of the solutions of the other system.

Morchalo [21] and Nohal [22] established asymptotic equivalence between linear integro-differential systems and their perturbations by using the dominated convergence theorem and the Hölder inequality. In [10] Choi et al. studied the asymptotic property of linear integro-differential systems by means of the resolvent matrices and useful equivalent systems. For

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asymptotic properties of linear Volterra difference systems we refer the reader to [8,9]. The uniform asymptotic stability of recurrent neural networks (RNNs) is analyzed by comparing RNNs to linear Volterra integro-differential systems in [19] and discrete analogs for a class of continuous-time recurrent neural networks are discussed in [20]. The results in this paper generalize some known properties concerning asymptotic equilibrium from the continuous and discrete cases [8–10] to the time scale situation.

Time scales theory was introduced by Hilger [14] to unify discrete and continuous differential calculus; see the books [4,5]. We refer the reader to [1–3,16,17] for results on Volterra and Fredholm type equations (both integral and integro-dynamic) on time scales. For example in [3] Adivar discusses the principle matrix and a variation of parameter formula. Lupulescu et al. [17] discussed the resolvent asymptotic stability, boundedness and show that the principle matrix and resolvent are equivalent for certain linear problems on time scales.

In this paper we assume the reader is familiar with the basic calculus of time scales. Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = \mathrm{col}(x_1, x_2, ...x_n)$ with a norm $\|\cdot\|$. We will use the same symbol $\|\cdot\|$ to denote the corresponding matrix norm in the space $M_n(\mathbb{R})$ of $n \times n$ matrices. We recall that $\|A\| := \sup\{\|Ax\|; \|x\| \leq 1\}$ and the following inequality $\|Ax\| \leq \|A\|\|x\|$ holds for all $A \in M_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. A time scale, denoted by $\mathbb{T}$, is an arbitrary, nonempty and closed subset of real numbers. The operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ called the forward jump operator is defined by $\sigma(t) := \inf\{s \in T, s > t\}$. The step size function $\mu\colon \mathbb{T} \to \mathbb{R}_+$ is given by $\mu(t) := \sigma(t) - t$. We say a point $t \in \mathbb{T}$ is right dense if $\mu(t) = 0$, and right scattered if $\mu(t) > 0$. Furthermore, a point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) := \sup\{s \in T, s < t\} = t$ and left scattered if $\rho(t) < t$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$. Throughout this work, we assume that $\sup \mathbb{T} = \infty$ with bounded graininess, i.e., $\mu(t) < \infty$. Moreover, the delta derivative of a function $f\colon \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) = \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

A function f is called rd-continuous provided that it is continuous at right dense points in $\mathbb{T}$, and has finite limit at left-dense points, and the set of rd-continuous functions are denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $C^1_{rd}(\mathbb{T}, \mathbb{R})$ includes the functions f whose derivative is in $C_{rd}(\mathbb{T}, \mathbb{R})$ too. For $s, t \in \mathbb{T}$ and a function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, the Δ-integral is defined to be

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s),$$

where $F \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ is an anti-derivative of f, i.e., $F^{\Delta} = f$ on $\mathbb{T}^k$. It should be noted that the Δ-integral by means of the Riemann sum is also introduced in [13].

Let $E \subseteq \mathbb{T}$ be a Δ-measurable set and let $p \in \mathbb{R}$ be such that $p \geq 1$ and let $f\colon E \to \mathbb{R}^n$ be a Δ-measurable function. We say f belongs to $L^p(E)$ provided that $\int_E \|f(t)\|^p \Delta t < \infty$.

For more details concerning L^p spaces we refer the reader to [23].

A function $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called regressive if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$, and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is called positively regressive if $1 + \mu(t)f(t) > 0$ on $\mathbb{T}^k$. The set of regressive functions and the set of positively regressive functions are denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\mathcal{R}^+(\mathbb{T}, \mathbb{R})$, respectively.

Let $f \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ and $s \in \mathbb{T}$, then the generalized exponential function $e_f(\cdot, s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the following initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t)x(t) \\ x(s) = 1. \end{cases}$$

For $h \in \mathbb{R}^+$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{Z}_h := \{z \in \mathbb{C} : -\pi/h < Im(z) \leq \pi/h\}$, and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h \in \mathbb{R}_0^+$ and $z \in \mathbb{C}_h$, the cylinder transformation $\xi_h \colon \mathbb{C}_h \to \mathbb{Z}_h$ is defined by

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \frac{1}{h}\operatorname{Log}(1 + zh), & h > 0, \end{cases}$$

and the exponential function can also be written in the form

$$e_f(t,s) := \exp\left\{ \int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau \right\} \quad \text{for } s,t \in \mathbb{T}.$$

For $f \in C_{rd}(\mathbb{T},\mathbb{R})$ and $\mu f^2 \in \mathcal{R}(\mathbb{T},\mathbb{R})$, the trigonometric functions $\cos_f$ and $\sin_f$ are defined by

$$\cos_f(t,s) = \frac{e_{if}(t,s) + e_{-if}(t,s)}{2} \quad \text{and} \quad \sin_f(t,s) = \frac{e_{if}(t,s) - e_{-if}(t,s)}{i2}.$$

For further details about these notions we refer the reader to [4,5].

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$, which is a complete metric space with the metric (distance) d defined by

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \text{for } (x_1,y_1),(x_2,y_2) \in \mathbb{T}_1 \times \mathbb{T}_2.$$

A function $f \colon \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is said to be continuous at $(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(x,y) - f(x_0,y_0)\| < \varepsilon$ for all $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying $d((x,y),(x_0,y_0)) < \delta$. If (x,y) is an isolated point of $\mathbb{T}_1 \times \mathbb{T}_2$, then the definition implies that every function $f \colon \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is continuous at (x,y). In particular, every function $f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z} \times \mathbb{Z}$.

Let $C_{rd}(\mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R})$ denote the set of functions $f(x,y)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the following properties:

(i) f is rd-continuous in x for fixed y;

(ii) f is rd-continuous in y for fixed x;

(iii) if $(x_0,y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_0 right-dense or maximal and y_0 right-dense or maximal, then f is continuous at (x_0,y_0);

(iv) if x_0 and y_0 are both left-dense, then the limit of $f(x,y)$ exists (finite) as (x,y) approaches (x_0,y_0) along any path in $\{(x,y) \in \mathbb{T}_1 \times \mathbb{T}_2 : x < x_0, y < y_0\}$.

A brief introduction into the two-variable time scales calculus can be found in [6].

Let us consider the Volterra integro-dynamic equation

$$y^{\Delta}(t) = A(t)y(t) + \int_{t_0}^{t} K(t,s)y(s)\Delta s + f(t), \qquad y(t_0) = y_0 \tag{1.1}$$

and the corresponding homogeneous equation

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s, \qquad x(t_0) = x_0, \tag{1.2}$$

where A is an $n \times n$ matrix function, f is a n-vector function, which are continuous on $\mathbb{T}_0 := \mathbb{T} \cap [0, \infty)$, and K is an $n \times n$ matrix function, which is continuous on $\Omega := \{(t,s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \le s \le t < \infty\}$.

Definition 1.1. The principle matrix solution of (1.2) is the $n \times n$ matrix function $Z(t,s)$ defined by

$$Z(t,s) := [x^1(t,s), x^2(t,s), \dots, x^n(t,s)],$$

where $x^i(t,s) (i = 1, 2, \dots, n)$ are the linearly independent solutions of (1.2). The principle matrix $Z(t,s)$ is called the transition matrix if $Z(\tau,\tau) = I$.

Therefore, the transition matrix of (1.2) at initial time τ is the unique solution of the matrix initial value problem

$$\begin{cases} Y^{\Delta}(t) = A(t)Y(t) + \int_{\tau}^{t} K(t,s)Y(s)\Delta s \\ Y(\tau) = I, \end{cases} \tag{1.3}$$

and $x(t) = Z(t,\tau)x_0$ is the unique solution of system (1.2).

The principle matrix is the unique solution of

$$\begin{aligned} &\Delta_t Z(t,s) = A(t)Z(t,s) - \int_{s}^{t} Z(t,\tau)K(\tau,s)\Delta\tau, \\ &Z(s,s) = I. \end{aligned} \tag{1.4}$$

Under continuity conditions on A and K, there is a unique solution of the initial value problem (see [17, Theorem 2.2])

$$\begin{aligned} &\Delta_s R(t,s) = -R(t,\sigma(s))A(s) - \int_{\sigma(s)}^{t} R(t,\sigma(\tau))K(\tau,s)\Delta\tau, \\ &R(t,t) = I. \end{aligned} \tag{1.5}$$

Both the principle matrix and the resolvent of the linear Volterra integro-dynamic equation are equivalent (see, [17, Theorem 2.7]). Then the unique solution $y(t,t_0,y_0)$ of (1.1) satisfying $y(t_0,t_0,y_0) = y_0$ is given by [3,17]

$$y(t,t_0,y_0) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau. \tag{1.6}$$

In the next section, we investigate the asymptotic property of (1.2) and its perturbation (1.1) by means of the resolvent matrix $R(t,s)$. With results concerning the asymptotic equilibrium we investigate asymptotic equivalence between two linear Volterra systems in Section 3. In the last section, we use a useful equivalent system from [17, Theorem 3.1] to study the asymptotic property of (1.1) and (1.2).

2 Asymptotic property

In this section we investigate the asymptotic property of the linear Volterra integro-dynamic system (1.1) and (1.2).

We need the following integral inequality.

Lemma 2.1. *Suppose that $u, f \in C_{rd}(\mathbb{T}, \mathbb{R})$ are nonnegative functions, and c is a nonnegative constant. Assume that $k(t,s)$ is a nonnegative and rd-continuous function for $s, t \in \mathbb{T}$ with $s \leq t$. Then*

$$u(t) \leq c + \int_{t_0}^{t} \left[f(s)u(s) + \int_{t_0}^{s} k(t,\tau)u(\tau)\Delta\tau \right] \Delta s \quad \text{for all } t \in \mathbb{T}_0$$

implies

$$u(t) \leq c e_p(t, t_0), \qquad t \in \mathbb{T}_0,$$

where $p(t) = f(t) + \int_{t_0}^{t} k(t,\tau)\Delta\tau$.

Proof. The proof is similar to [11, Theorem 3.13]. □

Let $p, v \colon \mathbb{T}_0 \to \mathbb{R}$ be nonnegative functions. The Hardy–Littlewood symbols O and o have the usual meaning: $z(n) = O(p(t))$ means that there exists $c > 0$ such that $\|z(t)\| \leq cp(t)$ for large t, and $z(t) = o(p(t))$ means that there exists $v(t)$ such that $\|z(t)\| \leq p(t)v(t)$ and $\lim_{t\to\infty} v(t) = 0$.

Definition 2.2. A linear Volterra integro-dynamic system (1.2) is said to have asymptotic equilibrium if there exist a unique $\zeta \in \mathbb{R}^n$ and $r > 0$ such that any solution $x(t)$ of (1.2) satisfies

$$x(t) = \zeta + o(1) \quad \text{as } t \to \infty \tag{2.1}$$

and conversely, for every $\zeta \in \mathbb{R}^n$ there exists a solution $x(t)$ of (1.2) with $\|x_0\| < r$ such that (2.1) is satisfied.

Our next result give necessary and sufficient conditions for (1.2) to have asymptotic equilibrium via the resolvent matrix $R(t,s)$.

Theorem 2.3. *System (1.2) has asymptotic equilibrium iff $\lim_{t\to\infty} R(t,t_0)$ exists and is invertible for each $t \geq t_0 \geq 0$.*

Proof. Suppose that (1.2) has asymptotic equilibrium. Then there exists a unique ζ and $r > 0$ such that if $x(t)$ is any solution of (1.2) with $\|x_0\| < r$ then $\lim_{t\to\infty} x(t) = \zeta$, i.e.,

$$\lim_{t\to\infty} R(t,t_0)x_0 = \zeta,$$

Then there exists $R_\infty(t_0)$ with $\lim_{t\to\infty} R(t,t_0) = R_\infty(t_0)$ for each $t_0 \geq 0$. Let $e_i = (0, \dots, 1, \dots, 0)^T$ be the unit vector in $\mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Then there exist solutions $x(t, t_0, x_{0i})$ of (1.2) such that

$$e_i = \lim_{t\to\infty} x(t, t_0, x_{0i}) = \lim_{t\to\infty} R(t,t_0)x_{0i} = R_\infty(t_0)x_{0i}, \qquad i = 1, 2, \dots, n.$$

It follows that

$$R_\infty(t_0)[x_{01} \dots x_{0n}] = I,$$

where $[x_{01} \dots x_{0n}]$ is the inverse matrix of $R_\infty(t_0)$. Thus $R_\infty(t_0)$ is invertible.

Conversely, let $\zeta \in \mathbb{R}^n$ be any vector. Then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $x_0 = R_\infty^{-1}(t_0)\zeta$ such that

$$\lim_{t\to\infty} x(t, t_0, x_0) = \lim_{t\to\infty} R(t, t_0)x_0 = \zeta.$$

This completes the proof. □

Corollary 2.4. *If* (1.2) *has asymptotic equilibrium, then there exists a positive constant* $M > 0$ *such that* $\|R(t,s)\| \le M$, *for* $0 \le t_0 \le s \le t$.

Theorem 2.5. *Assume that both* $A(t)$ *and* $\int_{t_0}^t K(t,s)\Delta s$ *belong to* $L^1(\mathbb{T}_0)$. *Then* (1.2) *has asymptotic equilibrium.*

Proof. Let $x(t)$ be the solution of (1.2). We can write (1.2) in an equivalent form

$$x(t) = x_0 + \int_{t_0}^t \left[A(s)x(s) + \int_{t_0}^s K(s,\tau)x(\tau)\Delta\tau \right] \Delta s. \tag{2.2}$$

Since $x(t) = R(t, t_0)x_0$ for each $x_0 \in \mathbb{R}^n$, it follows that

$$R(t, t_0) = I + \int_{t_0}^t \left[A(s)R(s, t_0) + \int_{t_0}^s K(s,\tau)R(\tau, t_0)\Delta\tau \right] \Delta s. \tag{2.3}$$

Let us take $u(t) = \|R(t, t_0)\|$ and

$$v(t) = 1 + \int_{t_0}^t \left[\|A(s)\| \, \|R(s, t_0)\| + \int_{t_0}^s \|K(s,\tau)\| \, \|R(\tau, t_0)\| \, \Delta\tau \right] \Delta s,$$

and we have the estimate

$$\begin{aligned} v(t) &= 1 + \int_{t_0}^t \left[\|A(s)\| \, u(s) + \int_{t_0}^s \|K(s,\tau)\| \, u(\tau)\Delta\tau \right] \Delta s \\ &\le 1 + \int_{t_0}^t \left[\|A(s)\| \, v(s) + \int_{t_0}^s \|K(s,\tau)\| \, v(\tau)\Delta\tau \right] \Delta s. \end{aligned}$$

Using Lemma 2.1, we obtain

$$v(t) \le e_p(t, t_0),$$

where $p(s) = \|A(s)\| + \int_{t_0}^s \|K(s,\tau)\| \, \Delta\tau$. Thus there exists a constant $M > 0$ with

$$v(t) \le e_p(\infty, t_0) < M.$$

It is easy to see that $u(t) \le v(t)$ for each $t \ge t_0$ and $v(t)$ is increasing and bounded. Furthermore, for any $t \ge t_1 \ge t_0$, we have

$$\begin{aligned} \|R(t, t_0) - R(t_1, t_0)\| &\le \int_{t_1}^t \left[\|A(s)\| \, \|R(s, t_0)\| + \int_{t_0}^s \|K(s,\tau)\| \, \|R(\tau, t_0)\| \, \Delta\tau \right] \Delta s \\ &= v(t) - v(t_1). \end{aligned}$$

This implies that, given any $\varepsilon > 0$, we can choose a $t_1 > 0$ sufficiently large so that

$$\|R(t, t_0) - R(t_1, t_0)\| < \varepsilon \quad \text{for all } t > t_1.$$

Hence $R(t, t_0)$ converges to a constant $n \times n$ matrix $R_\infty(t_0)$ as $t \to \infty$.

Next there exists a constant $N > 0$ such that $\|R(t,t_0)\| < N$ for each $t > t_0$. Since

$$\int_{t_0}^{\infty}\left[\|A(s)\| + \int_{t_0}^{s}\|K(s,\tau)\|\,\Delta\tau\right]\Delta s < \infty$$

then for a given $t_0 > 0$, we obtain

$$\int_{t_0}^{\infty}\left[\|A(s)\| + \int_{t_0}^{s}\|K(s,\tau)\|\,\Delta\tau\right]\Delta s < \frac{1}{N}. \tag{2.4}$$

Let us take

$$q(t,t_0) = \int_{t_0}^{t}\left[A(s)R(s,t_0) + \int_{t_0}^{s}K(s,\tau)R(\tau,t_0)\Delta\tau\right]\Delta s.$$

By taking norms, we have the estimate

$$\begin{aligned}\|q(t,t_0)\| &\le \int_{t_0}^{t}\left[\|A(s)\|\,\|R(s,t_0)\| + \int_{t_0}^{s}\|K(s,\tau)\|\,\|R(\tau,t_0)\|\,\Delta\tau\right]\Delta s.\\ &\le N\int_{t_0}^{t}\left[\|A(s)\| + \int_{t_0}^{s}\|K(s,\tau)\|\,\Delta\tau\right]\Delta s.\end{aligned}$$

Using (2.4), we obtain

$$\lim_{t\to\infty}\|q(t,t_0)\| < 1. \tag{2.5}$$

From (2.3) and (2.5), this implies that $\lim_{t\to\infty} R(t,t_0) = R_\infty$ is invertible. It follows from Theorem 2.3 that (1.2) has asymptotic equilibrium. □

Example 2.6. We consider the linear integro-dynamic equation

$$x^{\Delta}(t) = \frac{-1}{t\sigma(t)}x(t) - \int_{\frac{\pi}{2}}^{t} p\sin_p(t,\sigma(s))x(s)\Delta s, \qquad x\left(\frac{\pi}{2}\right) = 1, \tag{2.6}$$

where $A(t) = \frac{-1}{t\sigma(t)}$ and $K(t,s) = -p\sin_p(t,\sigma(s))$. Note that $(\frac{1}{t})^{\Delta} = \frac{-1}{t\sigma(t)}$ and $(\cos_p(t,s))^{\Delta} = -p\sin_p(t,\sigma(s))$ ([4, Lemma 3.26]). It is easy to see that $A(t)$ and $\int_{\frac{\pi}{2}}^{t} K(t,s)\Delta s$ belong to $L^1([\frac{\pi}{2},\infty)_{\mathbb{T}})$. From Theorem 2.5, the initial value problem (2.6) has asymptotic equilibrium.

Theorem 2.7. *Assume that* (1.2) *has asymptotic equilibrium and $f(t)$ belongs to $L^1(\mathbb{T}_0)$. Then* (1.1) *has asymptotic equilibrium.*

Proof. The solution $y(t)$ of (1.1), is given by

$$y(t) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau \quad \text{for each } t \ge t_0.$$

Let us consider $r(t) = \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau$. Since (1.2) has asymptotic equilibrium then by Corollary 2.4, $R(t,s)$ is bounded for $t_0 \le s \le t$ and $\int_{t_0}^{\infty}\|f(\tau)\|\,\Delta\tau < \infty$.Then there exists r_∞ with $\lim_{t\to\infty} r(t) = r_\infty$. This with Theorem 2.3 that $\lim_{t\to\infty} y(t) = x_i$ for some x_i in $\mathbb{R}^n$.

Conversely, let ξ be any vector $\mathbb{R}^n$ and consider $p(t) = \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau$. Since (1.2) has asymptotic equilibrium then again by Corollary 2.4 and $f(t)$ belongs to $L^1(\mathbb{T}_0)$ there exists p_∞ with $\lim_{t\to\infty} p(t) = p_\infty$. Thus there exists a solution $y(t)$ of (1.1) with initial value $y_0 = R_\infty^{-1}(\xi - p_\infty)$ such that

$$\begin{aligned} y(t) &= R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau \\ &= R(t,t_0)R_\infty^{-1}(\xi - p_\infty) + p_\infty - \int_t^\infty R(\infty,\sigma(\tau))f(\tau)\Delta\tau \\ &= \xi + o(1) \qquad \text{as } t\to\infty, \end{aligned}$$

since $\int_t^\infty R(\infty,\sigma(\tau))f(\tau)\Delta\tau \to 0$ as $t\to\infty$. □

Example 2.8. For $p\in\mathbb{R}$, such that $\ominus p = \frac{-p}{1+\mu(t)p} \in \mathcal{R}^+(\mathbb{T},\mathbb{R})$, we consider the linear integro-dynamic equation

$$x^\Delta(t) = \frac{-1}{t\sigma(t)}x(t) - \int_{\frac{\pi}{2}}^{t} p\sin_p(t,\sigma(s))x(s)\Delta s + e_{\ominus p}\left(\sigma(t),\frac{\pi}{2}\right), \qquad x\left(\frac{\pi}{2}\right) = 1, \tag{2.7}$$

where $A(t) = \frac{-1}{t\sigma(t)}$, $K(t,s) = -p\sin_p(t,\sigma(s))$ and $f(t) = e_{\ominus p}(\sigma(t),\frac{\pi}{2})$. Note that

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\infty} e_{\ominus p}\left(\sigma(t),\frac{\pi}{2}\right)\Delta t &= \lim_{b\to\infty}\frac{-1}{p}\int_{\frac{\pi}{2}}^{\infty}\frac{-p}{e_p(\sigma(t),\frac{\pi}{2})}\Delta t \\ &= \lim_{b\to\infty}\frac{-1}{p}\int_{\frac{\pi}{2}}^{\infty}\frac{-p}{e_p(\sigma(t),\frac{\pi}{2})}\Delta t \\ &= \lim_{b\to\infty}\frac{-1}{p}\int_{\frac{\pi}{2}}^{\infty}\left(\frac{1}{e_p(t,\frac{\pi}{2})}\right)^{\Delta}\Delta t \\ &= \lim_{b\to\infty}\frac{-1}{p}\left[\frac{1}{e_p(b,\frac{\pi}{2})} - 1\right] \\ &= \frac{1}{p}. \end{aligned}$$

It follows that $f(t)$ belongs to $L^1([\frac{\pi}{2},\infty)_{\mathbb{T}})$. From Theorem 2.7, (2.7) has asymptotic equilibrium.

Let us consider the Volterra integro-dynamic equation

$$y^\Delta(t) = A(t)y(t) + \int_{t_0}^{t} K(t,s)y(s)\Delta s + f(t), \tag{2.8}$$

and the corresponding homogeneous equation

$$x^\Delta(t) = A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\Delta s, \tag{2.9}$$

Definition 2.9. The two Volterra integro-dynamic systems (2.8) and (2.9) are said to be asymptotically equivalent if, for every solution $x(t)$ of (2.9), there exists a solution $y(t)$ of (2.8) such that

$$x(t) = y(t) + o(1) \quad \text{as } t\to\infty \tag{2.10}$$

and conversely, for every solution $y(t)$ of (2.8), there exists a solution $x(t)$ of (2.9) such that the asymptotic relationship (2.10) holds.

Next, we obtain asymptotic equivalence between (2.8) and (2.9).

Theorem 2.10. *Assume that* (2.9) *has asymptotic equilibrium and* $f(t)$ *belongs to* $L^1(\mathbb{T}_0)$. *Then* (2.8) *and* (2.9) *are asymptotically equivalent.*

Proof. Let $x(t)$ be the solution of (2.9) with the initial value x_0. Then there exists a solution $y(t)$ of (2.8) with initial condition $y(t_0) = x_0 - R_\infty^{-1} p_\infty$, such that

$$\begin{aligned} x(t) &= R(t,t_0)x_0 \\ &= y(t) + R(t,t_0)R_\infty^{-1}p_\infty - \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau \\ &= y(t) + o(1) \quad \text{ as } t \to \infty, \end{aligned}$$

where $p_\infty = \lim_{t\to\infty} \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau$.

Conversely, let $y(t)$ be the solution of (2.8) with the initial value y_0. Then there exists a solution $x(t)$ of (2.9) with initial condition $x(t_0) = y_0 + R_\infty^{-1}p_\infty$, such that

$$\begin{aligned} y(t) &= R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau \\ &= x(t) - R(t,t_0)R_\infty^{-1}p_\infty + \int_{t_0}^{t} R(t,\sigma(\tau))f(\tau)\Delta\tau \\ &= x(t) + o(1) \quad \text{ as } t \to \infty. \end{aligned}$$

This completes the proof. □

3 Asymptotic equivalence between two Volterra systems

Let us consider two linear Volterra integro-dynamic systems

$$x^\Delta = A(t)x + \int_{t_0}^{t} K(t,s)x(s)\Delta s, \qquad x(t_0) = x_0 \tag{3.1}$$

and

$$y^\Delta = C(t)y + \int_{t_0}^{t} D(t,s)y(s)\Delta s, \qquad y(t_0) = y_0 \tag{3.2}$$

(H1) Assume that $\int_{t_0}^{\infty} \|A(t) - C(t)\| \Delta t < \infty$ and $\int_{t_0}^{\infty} \|K(t,s) - D(t,s)\| \Delta s < \infty$ for almost all $t \in \mathbb{T}_0$.

Theorem 3.1. *Let (H1) hold. Then* (3.1) *has an asymptotic equilibrium if and only if* (3.2) *has an asymptotic equilibrium.*

Proof. Assume that (3.1) has an asymptotic equilibrium. We can write (3.2) in the form

$$y^\Delta = A(t)y + \int_{t_0}^{t} K(t,s)y(s)\Delta s - h(t,y(t)), \qquad y(t_0) = y_0,$$

where

$$h(t,y(t)) = [A(t) - C(t)]\, y(t) + \int_{t_0}^{t} [K(t,s) - D(t,s)]\, y(s)\Delta s.$$

Let $y(t)$ be any solution of (3.2) with the initial value $y(t_0) = y_0$. By using the variation of constants formula (1.6), we obtain

$$y(t) = R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(s))h(s,y(s))\Delta s. \tag{3.3}$$

It follows from (3.3) that

$$
\begin{aligned}
Q(t,t_0)y_0 &= R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(s))h(t,Q(s,t_0))\Delta s \\
&= R(t,t_0)y_0 + \int_{t_0}^{t} R(t,\sigma(s))\left(A(s)-C(s)\right)Q(s,t_0)y_0\Delta s \\
&\quad + \int_{t_0}^{t} R(t,\sigma(s))\left[\int_{t_0}^{s}\left(K(s,\tau)-D(s,\tau)\right)Q(\tau,t_0)y_0\Delta\tau\right]\Delta s,
\end{aligned}
$$

where $Q(t,s)$ is the unique solution of the initial value problem

$$
\begin{aligned}
&\Delta_s Q(t,s) = -Q(t,\sigma(s))C(s) - \int_{\sigma(s)}^{t} Q(t,\sigma(\tau))D(\tau,s)\Delta\tau, \\
&Q(t,t) = I.
\end{aligned}
$$

Also, it follows from the boundedness of $R(t,s)$ (with bound M) that

$$
\begin{aligned}
\|Q(t,t_0)\| \le{} & \|R(t,t_0)\| + \int_{t_0}^{t}\|R(t,\sigma(s))\|\Bigg[\|A(s)-C(s)\|\,\|Q(s,t_0)\| \\
& + \int_{t_0}^{s}\|K(s,\tau)-D(s,\tau)\|\,\|Q(\tau,t_0)\|\,\Delta\tau\Bigg]\Delta s \\
\le{} & M + M\int_{t_0}^{t}\left[\|A(s)-C(s)\|\,\|Q(s,t_0)\| + \int_{t_0}^{s}\|K(s,\tau)-D(s,\tau)\|\,\|Q(\tau,t_0)\|\,\Delta\tau\right]\Delta s.
\end{aligned}
$$

Putting $u(t) = \|Q(t,t_0)\|$ we obtain

$$
u(t) \le Me_q(t,t_0) \le Me_q(\infty,t_0) < \infty,
$$

where $q(t) = \|A(t)-C(t)\| + \int_{t_0}^{t}\|K(t,\tau)-D(t,\tau)\|\,\Delta\tau$. Thus

$$
\lim_{t\to\infty} Q(t,t_0) = Q_\infty(t_0)
$$

exists for each fixed $t_0 \in \mathbb{T}_0$.

Also we obtain the following relationship between $R(t,t_0)$ and $Q(t,t_0)$:

$$
\begin{aligned}
Q(t,t_0) &= R(t,t_0) + \int_{t_0}^{t} R(t,\sigma(s))\left(A(s)-C(s)\right)Q(s,t_0)\Delta s \\
&\quad + \int_{t_0}^{t} R(t,\sigma(s))\left[\int_{t_0}^{s}\left(K(s,\tau)-D(s,\tau)\right)Q(\tau,t_0)\Delta\tau\right]\Delta s \\
&= R(t,t_0) + R_\infty P(t,t_0),
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
P(t,t_0) &= R_\infty^{-1}\int_{t_0}^{t} R(t,\sigma(s))\left(A(s)-C(s)\right)Q(s,t_0)\Delta s \\
&\quad + R_\infty^{-1}\int_{t_0}^{t} R(t,\sigma(s))\left[\int_{t_0}^{s}\left(K(s,\tau)-D(s,\tau)\right)Q(\tau,t_0)\Delta\tau\right]\Delta s.
\end{aligned}
$$

Since both $R(t,t_0)$ and $Q(t,t_0)$ are bounded and (H1) holds, then $P(t,t_0)$ has the Cauchy property. Thus $\lim_{t\to\infty} P(t,t_0) = P_\infty(t_0)$ exists for each $t_0 \in [0,\infty)_{\mathbb{T}}$. We can choose $t_0 > 0$ sufficiently large so that $\|P_\infty(t_0)\| < 1$. Then we obtain from (3.4)

$$
Q_\infty = \lim_{t\to\infty} Q(t,t_0) = R_\infty[I + P_\infty(t_0)].
$$

It follows from $\|P_\infty(t_0)\| < 1$ that $I + P_\infty(t_0)$ is invertible and Q_∞ is also invertible. Hence (3.2) has an asymptotic equilibrium by Theorem 2.3.

In a similar manner we can obtain the converse. □

Our next result is about asymptotic equivalence between linear systems (3.1) and (3.2).

Theorem 3.2. *In addition to the assumptions of Theorem 3.1, suppose that (3.1) has an asymptotic equilibrium. Then (3.1) and (3.2) are asymptotically equivalent.*

Proof. We know that (3.2) has an asymptotic equilibrium by Theorem 3.1. Let $x(t, t_0, x_0)$ be any solution of (3.1). Then $\lim_{t\to\infty} x(t, t_0, x_0) = x_\infty$ exists. Thus there exists a solution $y(t, t_0, y_0)$ of (3.2) such that $\lim_{t\to\infty} y(t, t_0, y_0) = x_\infty$ and the asymptotic relationship

$$x(t, t_0, x_0) = y(t, t_0, y_0) + o(1) \quad \text{as } t \to \infty \tag{3.5}$$

holds. The converse asymptotic relationship can be obtained similarly. □

4 Asymptotic property via equivalent system

In this section we use a useful equivalent system to study the asymptotic property of (1.1) and (1.2).

Theorem 4.1. *Let $L(t, s)$ be an $n \times n$ continuously differentiable matrix function on Ω. Then (1.1) is equivalent to the following system*

$$\begin{cases} z^\Delta(t) = B(t)z(t) + L(t, t_0)x_0 + H(t), & t \in \mathbb{T}_0, \\ z(t_0) = x_0, \end{cases} \tag{4.1}$$

where

$$B(t) = A(t) - L(t, t) \quad \text{and} \quad H(t) = f(t) + \int_{t_0}^{t} L(t, \sigma(s))f(s)\Delta s, \tag{4.2}$$

and

$$K(t, s) + \Delta_s L(t, s) + L(t, \sigma(s))A(s) + \int_{\sigma(s)}^{t} L(t, \sigma(\tau))K(\tau, s)\Delta\tau = 0. \tag{4.3}$$

Proof. By taking $G(t, s) = 0$ in [17, Theorem 3.1], we obtain the result. □

The solution $z(t)$ of (4.1) with initial condition $z(t_0) = x_0$ is given by

$$z(t) = \Phi_B(t, t_0)x_0 + \int_{t_0}^{t} \Phi_B(t, \sigma(\tau)) \left[L(\tau, t_0)x_0 + H(\tau)\right] \Delta\tau, \tag{4.4}$$

where $\Phi_B(t, t_0)$ is a fundamental matrix solution of $z^\Delta(t) = B(t)z(t)$.

Our next theorem shows asymptotic equilibrium for the linear Volterra integro-dynamic system (1.1) by using the equivalent system (4.1) with $H(t) = 0$.

Theorem 4.2. *Let us assume that $\lim_{t\to\infty} \Phi_B(t, t_0) = \Phi_\infty$ is an invertible constant matrix and*

$$\int_{t_0}^{\infty} \|\Phi_B(t_0, \sigma(\tau))L(\tau, t_0)\| \Delta\tau < 1. \tag{4.5}$$

Then (1.1) with $f(t) = 0$ has an asymptotic equilibrium.

Proof. Let us consider an arbitrary $\zeta \in \mathbb{R}^n$. By using (4.5), it follows that

$$\int_{t_0}^{\infty} \Phi_B(t_0, \sigma(\tau)) L(\tau, t_0) \Delta\tau (= E)$$

exists, so $I + E$ is invertible. Thus we can find the unique solution x_0 of the linear system

$$\Phi_\infty (I + E) x_0 = \zeta$$

such that the solution of linear system is given by

$$x_0 = (I + E)^{-1} \Phi_\infty^{-1} \zeta. \tag{4.6}$$

Using (4.4), we obtain

$$\begin{aligned}
\lim_{t\to\infty} z(t) &= \lim_{t\to\infty} \left[\Phi_B(t, t_0) \left(I + \int_{t_0}^{t} \Phi_B(t_0, \sigma(\tau)) L(\tau, t_0) \right) x_0 \right] \Delta\tau \\
&= \Phi_\infty (I + E) x_0 \\
&= \Phi_\infty (I + E)(I + E)^{-1} \Phi_\infty^{-1} \zeta \\
&= \zeta.
\end{aligned}$$

Conversely, it is easy to see that the solution $z(t)$ of (4.1) tends to a vector $\zeta \in \mathbb{R}^n$ as $t \to \infty$. This completes the proof. □

Corollary 4.3. *In addition to the assumption of Theorem 4.2 suppose that* $\int_{t_0}^{\infty} \|H(\tau)\| \Delta\tau$ *exists. Then* (1.1) *has an asymptotic equilibrium.*

Proof. In the proof of Theorem 4.2 take

$$x_0 = (I + E)^{-1} \left[\Phi_\infty^{-1} \zeta - h_\infty \right],$$

where $h_\infty = \int_{t_0}^{\infty} \Phi_B(t_0, \sigma(\tau)) H(\tau) \Delta\tau$. Then the rest of the proof is the same as in Theorem 4.2. □

To obtain a sufficient condition on asymptotic equivalence between (1.1) and (1.2) we need the system

$$\begin{cases} u^\Delta(t) = B(t) u(t) + L(t, t_0) x_0, & t \in \mathbb{T}_0, \\ u(t_0) = x_0. \end{cases} \tag{4.7}$$

Theorem 4.4. *Assume that* $\lim_{t\to\infty} \Phi_B(t, t_0) = \Phi_\infty$ *and* $\int_{t_0}^{\infty} \Phi_B(t_0, \sigma(\tau)) H(\tau) \Delta\tau$ *exist. Then* (1.1) *and* (1.2) *are asymptotically equivalent.*

Proof. It suffices to prove that the systems (4.1) and (4.7) which are equivalent to (1.1) and (1.2) respectively, are asymptotically equivalent. Let $\tau(t)$ be any solution of (4.7) with the initial condition $u(t_0) = u_0$. Then the solution $z(t)$ of (4.1) is given by

$$\begin{aligned}
z(t) &= \Phi_B(t, t_0) x_0 + \int_{t_0}^{t} \Phi_B(t, \sigma(\tau)) \left[L(\tau, t_0) x_0 + H(\tau) \right] \Delta\tau \\
&= u(t) + \Phi_B(t, t_0)(x_0 - u_0) + \int_{t_0}^{t} \Phi_B(t, \sigma(\tau)) H(\tau) \Delta\tau.
\end{aligned}$$

Thus there exists a solution $z(t)$ of (4.1) with the initial value $x_0 = u_0 - h_\infty$ such that

$$\begin{aligned} z(t) &= u(t) + \Phi_B(t,t_0)\left[(x_0 - u_0) + h_\infty\right] \\ &= u(t) + o(1) \quad \text{as } t \to \infty, \end{aligned}$$

where, $h_\infty = \int_{t_0}^{\infty} \Phi_B(\infty,\sigma(\tau))H(\tau)\Delta\tau$.

Conversely, let $z(t)$ be any solution of (4.1). By taking $u_0 = x_0 + h_\infty$, there exists a solution $u(t)$ of (4.7) such that

$$\begin{aligned} z(t) &= u(t) + \Phi_B(t,t_0)\left[(h_\infty) + \tilde{h}(t)\right] \\ &= u(t) + o(1) \quad \text{as } t \to \infty, \end{aligned}$$

where $\tilde{h}(t) = \int_{t_0}^{t} \Phi_B(t_0,\sigma(\tau))H(\tau)\Delta\tau$. Hence (1.1) and (1.2) are asymptotically equivalent. This completes the proof. □

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